MATH 2060 TUTO 9

1. Determine the condition on the positive integers $m, n \geq 1$,if and only if

$$f(x) = \begin{cases} x^m \cos \frac{1}{x^n}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuously differentiable.

First find condition for f diff.

Consider $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^m \cos\frac{1}{x^n} - 0}{x} = \lim_{x\to 0} x^{m-1} \cos\frac{1}{x^n}$

For $m \geq 2$, $\left| x^{m-1} \cos\frac{1}{x^n} \right| \leq |x|^{m-1}$ and $\lim_{x\to 0} |x|^{m-1}$

$\Rightarrow$ $f'(0) = 0$ exists by squeeze thm

For $m=1$, $\lim_{x\to 0} \cos\frac{1}{x^n}$ does not exist

$\Rightarrow$ $f'(0)$ does not exist.

So f is diff. $\iff$ $m \geq 2$.

Next find condition for f' cts.

Note $f'(x) = \begin{cases} m x^{m-1} \cos\frac{1}{x^n} + n x^{m-n-1} \sin\frac{1}{x^n}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly f' is cts for $x \neq 0$.

And $\lim_{x\to 0} m x^{m-1} \cos\frac{1}{x^n} = 0$ as $m-1 \geq 1 \iff m \geq 2$

So f' is cts at $x=0$ $\iff$ $\lim_{x\to 0} x^{m-n-1} \sin\frac{1}{x^n} = 0$

$\iff$ $m-n-1 \geq 1$

$\iff$ $m \geq n+2$

(since $\left|\sin\frac{1}{x^n}\right| \leq 1$ and $\lim_{x\to 0} \sin\frac{1}{x^n}$ DNE)

Since $n \geq 1$, $m \geq n+2 \Rightarrow m \geq 3 > 2$ (so f' exists)

Hence f is C^1 $\iff$ $m \geq n+2$ //

2. Suppose that f is continuous on $[a, b]$ and f'' exists on (a, b). Suppose that the graph of f and the line segment joining the end points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ with $x_0 \in (a, b)$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

(Done in TUT03).

Apply MVT to f on $[a, x_0]$, $\exists\, c_1 \in (a, x_0)$ s.t.

$$f(x_0) - f(a) = f'(c_1)(x_0 - a)$$

Apply MVT to f on $[x_0, b]$, $\exists\, c_2 \in (x_0, b)$ s.t.

$$f(b) - f(x_0) = f'(c_2)(b - x_0)$$

$$\Rightarrow \begin{cases} f'(c_1) = \dfrac{f(x_0) - f(a)}{x_0 - a} = \text{slope of the line segment} \\[2ex] f'(c_2) = \dfrac{f(b) - f(x_0)}{b - x_0} = \text{slope of the line segment} \end{cases}$$

$$\Rightarrow \quad f'(c_1) = f'(c_2)$$

Note $a < c_1 < x_0 < c_2 < b$, and

f'' exists on (a, b) $\Rightarrow$ f' cts and diff. on $[c_1, c_2]$.

Apply MVT to f' on $[c_1, c_2]$, $\exists\, c \in (c_1, c_2)$ s.t.

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = 0$$

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3. Let $R_n(x)$ be the remainder of the n-th Taylor polynomial of $(1+x)^{\frac{1}{n}}$ at $x=0$, where $n \geq 2$ is an integer. Show that for any $x > 0$,

$$n \text{ even} \implies R_n(x) \leq \frac{(n-1)(2n-1)(3n-1)\cdots(n^2-1)}{(n+1)!n^{n+1}}x^{n+1}$$

$$n \text{ odd} \implies R_n(x) \geq -\frac{(n-1)(2n-1)(3n-1)\cdots(n^2-1)}{(n+1)!n^{n+1}}x^{n+1}.$$

Let $f(x) = (1+x)^{\frac{1}{n}}$

Recall: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$ $\exists\, c \in (0,x)$

1) Compute $f^{(n+1)}(x)$.

$f'(x) = \frac{1}{n}(1+x)^{\frac{1}{n}-1}$

$f''(x) = \frac{1}{n}\left(\frac{1}{n}-1\right)(1+x)^{\frac{1}{n}-2}$

$\vdots$

$f^{(n+1)}(x) = \frac{1}{n}\left(\frac{1}{n}-1\right)\cdots\left(\frac{1}{n}-n\right)(1+x)^{\frac{1}{n}-n-1}$

2) $R_n(x)$

By Taylor's Thm, $\forall x > 0$, $\exists\, c \in (0,x)$ s.t.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

$$= \frac{1}{(n+1)!}\underbrace{\frac{1}{n}\left(\frac{1}{n}-1\right)\cdots\left(\frac{1}{n}-n\right)}_{n+1 \text{ terms}}(1+c)^{\frac{1}{n}-n-1}x^{n+1}$$

$$= \frac{1}{(n+1)!}\cdot\frac{1}{n^{n+1}}\underbrace{(1-n)(1-2n)\cdots(1-n^2)}_{n \text{ terms}}\frac{1}{(1+c)^{n+1-\frac{1}{n}}}x^{n+1}$$

$$= \frac{(-1)^n(n-1)(2n-1)\cdots(n^2-1)}{(n+1)!\,n^{n+1}}\frac{1}{(1+c)^{n+1-\frac{1}{n}}}x^{n+1}$$

If n is even, then

$$R_n(x) = \frac{(n-1)(2n-1)\cdots(n^2-1)}{(n+1)!\,n^{n+1}}\frac{1}{(1+c)^{n+1-\frac{1}{n}}}x^{n+1}$$

≤ 1 ?

Since $0 < c < x$, $0 < \frac{1}{1+c} < 1$, $n+1-\frac{1}{n} > 0$ and all other factors are +ve, we have

$$R_n(x) \leq \frac{(n-1)(2n-1)\cdots(n^2-1)}{(n+1)!\, n^{n+1}} x^{n+1}$$

If n is odd, then

$$R_n(x) = -\frac{(n-1)(2n-1)\cdots(n^2-1)}{(n+1)!\, n^{n+1}} \frac{1}{(1+c)^{n+1-\frac{1}{n}}} x^{n+1}$$

$$\geq -\frac{(n-1)(2n-1)\cdots(n^2-1)}{(n+1)!\, n^{n+1}} x^{n+1}$$

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4. Using definition, show that

$$G(x) := \begin{cases} \frac{1}{n}, & \text{if } x = 1 - \frac{1}{n} \quad (n = 1, 2, \dots) \\ 0, & \text{elsewhere on } [0,1]. \end{cases}$$

is (Riemann) integrable on $[0,1]$ and $\int_0^1 G = 0$.

Let $\varepsilon > 0$. Set $E_\varepsilon := \{x \in [0,1] : G(x) \geq \varepsilon\}$

$= \{0, \frac{1}{2}, \frac{2}{3}, \dots, 1 - \frac{1}{N_\varepsilon}\}$ is a finite set.

where $N_\varepsilon = [1/\varepsilon]$ the largest integer $\leq 1/\varepsilon$

($N_\varepsilon \leq 1/\varepsilon$; $N_\varepsilon + 1 > 1/\varepsilon$

$\Rightarrow \frac{1}{N_\varepsilon} \geq \varepsilon$; $\frac{1}{N_\varepsilon + 1} < \varepsilon$)

Take $\delta := \frac{\varepsilon}{2N_\varepsilon + 1} > 0$

If $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a tagged partition of $[0,1]$ with $\|\dot{P}\| < \delta$,

then $S(G; \dot{P}) = \sum_{i=1}^n G(t_i)(x_i - x_{i-1})$

$$= \underbrace{\sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^n G(t_i)(x_i - x_{i-1})}_{\text{I}} + \underbrace{\sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^n G(t_i)(x_i - x_{i-1})}_{\text{II}}$$

I: $t_i \notin E_\varepsilon \Rightarrow 0 \leq G(t_i) < \varepsilon$

So $0 \leq \sum_{\substack{i=1 \\ t_i \notin E_\varepsilon}}^n G(t_i)(x_i - x_{i-1}) < \varepsilon \sum_{i=1}^n (x_i - x_{i-1}) = \varepsilon$

II: $0 \leq G(x) \leq 1$ and each tag belongs to at most 2 subintervals

So $0 \leq \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^n G(t_i)(x_i - x_{i-1}) \leq \sum_{\substack{i=1 \\ t_i \in E_\varepsilon}}^n (1)(\delta) \leq \delta(2N_\varepsilon) < \varepsilon$

at most $2N_\varepsilon$ terms

Hence $0 \leq S(G; \dot{P}) < \varepsilon + \varepsilon = 2\varepsilon$ for any $\dot{P}$ with $\|\dot{P}\| < \delta$

Since $\varepsilon > 0$ is arbitrary, $G \in \mathcal{R}[0,1]$ and $\int_0^1 G = 0$ //

5. Using squeeze theorem, discuss the integrability of

$$f(x) = \begin{cases} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

on the interval $[0,1]$.

(Did a similar question in TUTO 6)

Let $\varepsilon \in (0,1)$. Define

$$\alpha_\varepsilon(x) := \begin{cases} -1, & x \in [0, \varepsilon/4) \\ f(x), & x \in [\varepsilon/4, 1] \end{cases}$$

$$\omega_\varepsilon(x) := \begin{cases} 1, & x \in [0, \varepsilon/4) \\ f(x), & x \in [\varepsilon/4, 1] \end{cases}$$

Then $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \forall x \in [0,1]$ (since $|f| \leq 1$)

Need to check: $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b]$ and $\int (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

Note α_ε is constant on $[0, \varepsilon/4]$ except at $x = \varepsilon/4$

$\Rightarrow$ α_ε integrable on $[0, \varepsilon/4]$ by Thm 7.1.3

Also, $f = \cos(\frac{1}{x})$ cts on $[\varepsilon/4, 1] \Rightarrow f$ integrable on $[\varepsilon/4, 1]$

So α_ε integrable on $[\varepsilon/4, 1]$

By Additivity Thm 7.2.9, $\alpha_\varepsilon \in \mathcal{R}[0,1]$.

Similarly, $\omega_\varepsilon \in \mathcal{R}[0,1]$.

Moreover,
$$\int_0^1 (\omega_\varepsilon - \alpha_\varepsilon) = \int_0^{\varepsilon/4} (\omega_\varepsilon - \alpha_\varepsilon) + \int_{\varepsilon/4}^1 (\omega_\varepsilon - \alpha_\varepsilon)$$
$$= \int_0^{\varepsilon/4} 2 + \int_{\varepsilon/4}^1 0$$
$$= \varepsilon/2 < \varepsilon$$

By squeeze thm $f \in \mathcal{R}[0,1]$ //